Inverse branching rules

1989 J. Phys. A: Math. Gen. 22 L1055

(http://iopscience.iop.org/0305-4470/22/22/002)

J. Phys. A: Math. Gen. **22** (1989) L1055–L1060. Printed in the UK

LETTER TO THE EDITOR

Inverse branching rules

C J Cummins

Department of Mathematics, Concordia University, Sir George Williams Campus, 1455 De Maisonneuve Blvd West, Montréal, Québec, Canada†

Received 14 August 1989

Abstract. A method is given for 'inverting' branching rules which is based on a technique for computing Kronecker products due to King. This approach simplifies the calculation of plethysms and branching rules of representations of exceptional groups. It may also be applied to basic spin plethysms.

It is well known that the Kronecker product of two irreducible representations of a compact, simple Lie group G with highest-weight labels λ_G and μ_G may be computed using the Racah–Speiser (1964) formula. The only information required is the weight-space decomposition of one of the representations, μ_G say, and the twisted Weyl group action, $\lambda_G \to w(\lambda_G + \delta_G) - \delta_G$, on weight-space. This formula may be stated in the form of an algorithm:

Algorithm 1 (Racah 1964, Speiser 1964)

Input:

G, λ_G, μ_G	simple Lie group G: two irreducible representations
$\Sigma_{\sigma_T} M^{\sigma_T}_{\mu_G} \sigma_T$	weight-space decomposition of the representation μ_G

Output:

$K^{\rho_G}_{\lambda_G \mu_G}$	multiplicity of the representation ρ_G in the Kronecker product of the representations λ_G and μ_G

Method:

1.1. Add the highest-weight label of the representation λ_G to each of the weight labels in $\Sigma_{\sigma_H} M^{\sigma_H}_{\mu_G} \sigma_H$.

1.2. Apply the G weight-space modification rules (i.e. apply the Weyl group with twisted action) to make dominant the resulting labels. This produces a sum of representations of G which is the required decomposition of the Kronecker product.

Note that the weight-space modifications may result in representations with negative or zero multiplicities. The latter may be discarded, while the former 'cancel' terms with positive multiplicities to yield, eventually, non-negative multiplicities for all representations; this process is best understood in terms of manipulations of the corresponding characters.

† e-mail: CUMMINS@vax2.concordia.CA

0305-4470/89/221055+06$02.50

King (1981) observed that the weight-space decomposition of μ_G may be considered to be a branching of this representation to a maximal, toroidal subgroup T of G, and that the combination of weight labels in step 1.1 is simply the Kronecker product of representations of T—the weight labels being labels of one-dimensional, irreducible representations of T. This suggests the following generalisation of the Racah–Speiser formula based on the branching of μ_G to an equal-rank, regular, reductive subgroup H of G:

Algorithm 2 (King 1981; see also Black et al 1983)

Input:

G, H	simple Lie group G: equal-rank, regular, reductive subgroup H
λ_G, μ_G	two irreducible representations of G
$\Sigma_{\sigma_H} M^{\sigma_H}_{\mu_G}\sigma_H$	branching of the representation μ_G to H

Output:

$K^{\rho_G}_{\lambda_G\mu_G}$	multiplicity of the representation ρ_G in the Kronecker product of the representations λ_G and μ_G

Method:

2.1. Add $\delta_G - \delta_H$ to the highest-weight label of λ_G, where δ_G is half the sum of the positive roots of G and δ_H is half the sum of the positive roots of H.

2.2. Take the Kronecker product of the corresponding representation $(\lambda_G + \delta_G - \delta_H)_H$ of H with $\Sigma_{\sigma_H} M^{\sigma_H}_{\mu_G}\sigma_H$ in H and express the result as a sum of irreducible representations of H.

2.3. Subtract $\delta_G - \delta_H$ from each of the highest-weight labels in this sum and apply the G weight-space modification rules as in step 1.2 above (see Black *et al* 1983, table 4) to produce a sum of representations of G which is the required decomposition of the Kronecker product.

When using this algorithm it is convenient to label the representations of G in a scheme adapted to the subgroup H: see King and Al-Qubanchi (1981), where weight-space modification rules are also given for many exceptional-group subgroup pairs. If H is chosen to be a unitary group then there exists an extremely efficient algorithm—the Littlewood–Richardson rule—for computing the required Kronecker products. In addition the number of modifications necessary is typically much smaller than the number required for the Racah–Speiser formula. Consequently the algorithm is very quick once the initial branching is known. See section 8 of Black *et al* (1983) for some examples.

The aim of this letter is to point out that if we specialise to the case where the representation λ_G is the identity representation of G, then this algorithm may be used to 'invert' branchings from G to H in a very efficient manner. To be precise we have the following algorithm.

Algorithm 3

Input:

G, H	simple Lie group G: equal-rank, regular, reductive subgroup H
$\Sigma_{\sigma_H} M^{\sigma_H}\sigma_H$	a sum of irreducible representations, which is known to be the restriction to H of some representation of G

Output:

$\sum_{\rho_G} K^{\rho_G}\rho_G$ decomposition into irreducibles of the (unique) representation of G which branches to $\sum_{\sigma_H} M^{\sigma_H}\sigma_H$

Method:

3.1. Take the Kronecker product of the representation $(\delta_G-\delta_H)_H$ of H with $\Sigma_{\sigma_H} M^{\sigma_H}\sigma_H$ in H and express the result as a sum of irreducible representations of H.

3.2. Apply step 2.3 of the previous algorithm. The result is $\Sigma_{\rho_G} K^{\rho_G}\rho_G$.

This algorithm is simply algorithm 2 applied to the Kronecker product of the identity representation of G and $\Sigma_{\rho_G} K^{\rho_G}\rho_G$. Since algorithm 2 does not actually require this representation as input, but rather its branching to H, $\Sigma_{\sigma_H} M^{\sigma_H}\sigma_H$, we may apply it to calculate the product. On the other hand we know that the result must be $\Sigma_{\rho_G} K^{\rho_G}\rho_G$, which is the representation of G we wish to calculate. Hopefully the following three examples will convince the reader that this algorithm is useful.

Example 1. Plethysms in E_7. To find $(1^2)^{\otimes\{2\}}$, the symmetric square of the 56-dimensional representation of E_7, which is labelled (1^2) in an SU_8 basis (cf section 13 of Wybourne and Bowick (1977) where a slightly different labelling scheme is used).

(1) Under the restriction $E_7\downarrow SU_8$ we have the branching $(1^2)\downarrow\{\bar{1}^2\}+\{1^2\}$.

(2) Using standard techniques (see for example appendix 1 of Cummins and King (1986) and its references) in SU_8:

$$(\{\bar{1}^2\}+\{1^2\})^{\otimes\{1^2\}}=\{\bar{2}^2\}+\{\bar{1}^4\}+\{\bar{1}^2;1^2\}+\{\bar{1};1\}+\{0\}+\{2^2\}+\{1^4\}.$$

In terms of covariant representation labels this is

$$\{2^6\}+\{2^2\}+2\{1^4\}+\{2^21^4\}+\{21^6\}+\{0\}.$$

(3) This last sum is the restriction to SU_8 of the required E_7 representation, so algorithm 3 is applicable. In this case $(\delta_G-\delta_H)_H$ is the representation $\{10\}$ of SU_8, but as explained in section 8 of Black *et al* (1983) it is sufficient in this case to use the representation $\{2\}$.

(4) Using the Littlewood–Richardson rule:

$$\{2^6\}\cdot\{2\}=\{42^5\}+\{32^51\}+\{2^7\}$$
$$\{2^2\}\cdot\{2\}=\{42\}+\{321\}+\{2^3\}$$
$$2\{1^4\}\cdot\{2\}=2\{31^3\}+2\{21^4\}$$
$$\{2^21^4\}\cdot\{2\}=\{421^4\}+\{32^21^3\}+\{321^5\}+\{2^31^4\}$$
$$\{21^6\}\cdot\{2\}=\{41^6\}+\{321^5\}+\{31^7\}+\{2^21^6\}$$
$$\{0\}\cdot\{2\}=\{2\}.$$

(5) Subtracting 2 from each first representation label we obtain the non-standard E_7 representation labels:

$$(2^6)+(12^51)+(02^6)$$
$$(2^2)+(121)+(02^2)$$
$$2(1^4)+2(01^4)$$
$$(2^21^4)+(12^21^3)+(121^5)+(02^21^4)$$
$$(21^6)+(121^5)+(1^8)+(021^6).$$

Of these representations (2^2), (21^6) and (0) are standard in E_7, while (1^8) modifies to (0) and $2(01^4)$ modifies to $-2(0)$, which cancels the other two copies of the identity representation. All the other representations modify to zero and thus (cf equation (56) of Wybourne and Bowick (1977))

$$(1^2)^{\otimes\{2\}}=(2^2)+(21^6).$$

This technique should considerably increase the range of exceptional group plethysms that can be calculated, since previous calculations have required tabulations of branching rules for the final 'inversion', or have used weight-space methods.

Example 2. Basic spin plethysms. We may calculate basic spin plethysms in a similar manner. Consider, for example, $\Delta_-^{\otimes\{2\}}$, the symmetric square of one of the four-dimensional spin representations of SO_6:

(1) Under the restriction $SO_6\downarrow U_3$,

$$\Delta_-\downarrow\varepsilon^{-1/2}(\{1^2\}+\{0\}).$$

Here ε is the determinantal representation of U_3.

(2) Again using standard techniques we find in U_3:

$$\begin{aligned}(\varepsilon^{-1/2}(\{1^2\}+\{0\}))^{\otimes\{2\}}&=\{\bar{1};\,1^2\}+\{\bar{1}\}+\{\bar{1}^3\}\\&=\{1,1,-1\}+\{0,0,-1\}+\{-1,-1,-1\}.\end{aligned}$$

(3) Now apply algorithm 3; in this case $(\delta_G-\delta_H)_H$ is $\{0\}$, so addition and subtraction of the corresponding highest-weight label are null operations. Hence we obtain the non-standard SO_6 representations:

$$[1,1,-1]+[0,0,-1]+[-1,-1,-1].$$

Of these the first, $[1^3]_-$, is standard, while the second two modify to zero. So the required plethysm is:

$$\Delta_-^{\otimes\{2\}}=[1^3]_-.$$

King *et al* (1981) have given explicit formulae for basic spin plethysms up to the third degree and a prescription for the calculation of fourth-degree plethysms. In addition all cases up to SO_8 can be analysed by well-known automorphisms and isomorphisms. The method just outlined would be of use for computing plethysms of higher degree and for higher-rank groups in a uniform manner suitable for computer implementation.

Example 3. Branchings in E_8. In Wybourne (1984) branching rules for $E_8\downarrow SO_{16}$ are computed using the subgroup chains, $E_8\downarrow SU_2\times E_7\downarrow U_1\times SU_8$ and $SO_{16}\downarrow U_1\times SU_8$†. Using algorithm 3 we may simplify the last step of this procedure. For example to calculate the branching of the 248-dimensional representation of E_8, labelled by (1^2) in a SO_{16} basis:

(1) Under the restriction $E_8\downarrow SU_2\times E_7$

$$(1^2)\downarrow\{2\}\times(0)+\{1\}\times(1^2)+\{0\}\times(21^6)$$

and under the further restriction, $SU_2\times E_7\downarrow U_1\times SU_8$, this branches to

$$(2+0+\bar{2})\times\{0\}+(1+\bar{1})\times\{\bar{1}^2+1^2\}+(0)\times\{\bar{1};\,1+1^4\}.$$

† Actually a litte more care is required here. The centres of SU_2 and E_7 should be identified, so $SU_2\times E_7$ is a central product. This means that $U_1\times SU_8$ is a covering group of U_8 which is evident, since multiple-valued representations of U_8 arise. Similarly the 'SO_6' and 'U_3' of example 2 are covering groups.

(2) This last sum of representations is the branching of the required representation of SO_{16} under the restriction $SO_{16}\downarrow U_1\times SU_8$. Before applying algorithm 3 it is necessary first to convert the $U_1\times SU_8$ labels to U_8 labels. To do this we calculate $e=(a-b)/8$ where a is twice the U_1 label and b the weight of the SU_8 label. This number is then added to each part of the SU_8 label:

$$(2)\times\{0\}\to(\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12)$$
$$(0)\times\{0\}\to(0,0,0,0,0,0,0,0)$$
$$(\bar2)\times\{0\}\to(\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2})$$
$$(1)\times\{\bar1^2\}\to(\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac{\bar1}{2},\tfrac{\bar1}{2})$$
$$(\bar1)\times\{\bar1^2\}\to(0,0,0,0,0,0,\bar1,\bar1)$$
$$(1)\times\{1^2\}\to(1,1,0,0,0,0,0,0)$$
$$(\bar1)\times\{1^2\}\to(\tfrac12,\tfrac12,\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2})$$
$$(0)\times\{\bar1;1\}\to(1,0,0,0,0,0,0,\bar1)$$
$$(0)\times\{1^4\}\to(\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2},\tfrac{\bar1}{2}).$$

(3) We now continue as in example 2; only modifications in SO_{16} are necessary. The first, second and sixth terms are standard, while the fifth modifies to $-(0,0,0,0,0,0,0,0)$ and so cancels the second. The rest modify to zero. The final answer is thus:

$$(\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12,\tfrac12)+(1,1,0,0,0,0,0,0)$$

which, in more usual notation, is

$$\Delta_+ + [1^2].$$

In this example we could have reversed the procedure to deduce the $SU_2\times E_7$ content of (1^2) given its branching to SO_{16}, since no E_7 products are required. This technique would in general allow one to calculate branchings to equal-rank, regular subgroups once the branching to one such subgroup is known and provided a suitable common subgroup could be found.

Remarks

(1) Although algorithm 3 allows us to 'invert' branchings from G to H it does not seem possible, in general, to deduce from this a way of computing the branchings from G to H.

(2) It would be interesting to see if similar techniques could be applied to Kac–Moody algebras. Many of the details in this case, however, have yet to be worked out.

(3) As the examples show, it is not unusual for a large number of representations of H to be produced that do not contribute to the final answer. An advantage of algorithm 3 is that these representations may be modified, and if necessary discarded, as produced.

I would like to thank Professors R C King and B G Wybourne for encouragement and useful conversations. Thanks are also due to Professors D J Rowe and B Castel who organised the CAP-NSERC 1989 Summer Institute in Theoretical Physics at Kingston, Ontario, where some of this work was done. This work was partially supported by the Natural Sciences and Engineering Council of Canada.

References

Black G R E, King R C and Wybourne B G 1983 Kronecker products for compact semisimple Lie groups *J. Phys. A: Math. Gen.* **16** 1555-89

Cummins C J and King R C 1986 Absolute minima of the Higgs potential for the 75 of SU(5) *J. Phys. A: Math. Gen.* **19** 161-81

King R C 1981 Kronecker products of representations of exceptional Lie groups *J. Phys. A: Math. Gen.* **14** 77-83

King R C and Al-Qubanchi A H A 1981 Natural labelling schemes for simple roots and irreducible representations of exceptional Lie algebras *J. Phys. A: Math. Gen.* **14** 15-49

King R C, Luan-Dehuai and Wybourne B G 1981 Symmetrised powers of rotation group representations *J. Phys. A: Math. Gen.* **14** 2509-38

Racah G 1964 *Group Theoretical Concepts and Methods in Elementary particle Physics* ed F Gursey (New York: Gordon and Breach) pp 1-36

Speiser D 1964 *Group Theoretical Concepts and Methods in Elementary Particle Physics* ed F Gursey (New York: Gordon and Breach) pp 237-46

Wybourne B G 1984 Branching rules for $E_8{\downarrow}SO_{16}$ *J. Phys. A: Math. Gen.* **17** 1397-402

Wybourne B G and Bowick M J 1977 Basic properties of the exceptional Lie groups *Aust. J. Phys.* **30** 259-86